

Complex interpolation of weighted noncommutative L_p -spaces

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Abstract

Let \mathcal{M} be a semifinite von Neumann algebra equipped with a semifinite normal faithful trace τ . Let d be an injective positive measurable operator with respect to (\mathcal{M}, τ) such that d^{-1} is also measurable. Define

$$L_p(d) = \{x \in L_0(\mathcal{M}) : dx + xd \in L_p(\mathcal{M})\} \quad \text{and} \quad \|x\|_{L_p(d)} = \|dx + xd\|_p.$$

We show that for $1 \leq p_0 < p_1 \leq \infty$, $0 < \theta < 1$ and $\alpha_0 \geq 0, \alpha_1 \geq 0$ the interpolation equality

$$(L_{p_0}(d^{\alpha_0}), L_{p_1}(d^{\alpha_1}))_\theta = L_p(d^\alpha)$$

holds with equivalent norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

1 Introduction

Let \mathcal{M} be a semifinite von Neumann algebra equipped with a normal faithful semifinite trace τ . For $1 \leq p \leq \infty$, let $L_p(\mathcal{M})$ denote the noncommutative L_p -space associated with (\mathcal{M}, τ) . The norm of $L_p(\mathcal{M})$ is denoted by $\|\cdot\|_p$. All spaces $L_p(\mathcal{M})$ are continuously injected into the topological involutive algebra $L_0(\mathcal{M})$ of measurable operators with respect to (\mathcal{M}, τ) . This injection turns $(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))$ into a compatible couple. We then have the following well-known identity on the complex interpolation of noncommutative L_p -spaces: for any $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$,

$$(1.1) \quad (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta = L_p(\mathcal{M})$$

with equal norms, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We refer to [4], [10], [15] and [14] for semifinite noncommutative L_p -spaces and to [2] for interpolation.

The aim of this note is to consider the weighted version of (1.1). Let $d \in L_0(\mathcal{M})$ be a positive injective operator such that $d^{-1} \in L_0(\mathcal{M})$. We will call d a density. Define

$$L_p^r(d) = \{x \in L_0(\mathcal{M}) : xd \in L_p(\mathcal{M})\}$$

equipped with the norm

$$\|x\|_{L_p^r(d)} = \|xd\|_p.$$

Then by standard arguments one easily deduces from (1.1) the following right weighted analogue. Let θ, p_0, p_1, p be as in (1.1), and let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then

$$(1.2) \quad (L_{p_0}^r(d^{\alpha_0}), L_{p_1}^r(d^{\alpha_1}))_\theta = L_p^r(d^\alpha)$$

with equal norms. One has, of course, a similar equality for the left weighted spaces. However, the matter becomes highly subtle when one considers the sum of two multiplication maps, one from left and another from right. Thus let

$$L_p(d) = \{x \in L_0(\mathcal{M}) : dx + xd \in L_p(\mathcal{M})\}$$

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equipped with the norm

$$\|x\|_{L_p(d)} = \|dx + xd\|_p.$$

We will see later that $L_p(d)$ is complete, so is a Banach space for any $1 \leq p \leq \infty$. The compatibility on these weighted spaces is induced by the identity of $L_0(\mathcal{M})$. The following is the two-sided version of (1.2), which is the main result of this note.

Theorem 1.1 *Let $0 < \theta < 1$, $1 \leq p_0, p_1 \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Assume $1 < p < \infty$. Let $\alpha_0, \alpha_1 \geq 0$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(1.3) \quad (L_{p_0}(d^{\alpha_0}), L_{p_1}(d^{\alpha_1}))_\theta = L_p(d^\alpha)$$

with equivalent norms.

It is worth to note that while (1.2) holds for any real α_0 and α_1 , (1.3) may fail when α_0 and α_1 are of opposite signs (see Remark 3.8 below).

The theorem above is closely related to a recent remarkable interpolation theorem of Junge and Parcet [8]. Define

$$\Delta_p(d) = \{x \in L_0(\mathcal{M}) : dx, xd \in L_p(\mathcal{M})\}$$

and

$$\|x\|_{\Delta_p(d)} = \max(\|dx\|_p, \|xd\|_p).$$

Keeping the assumptions of Theorem 1.1 on the indices, we have

$$(\Delta_{p_0}(d^{\alpha_0}), \Delta_{p_1}(d^{\alpha_1}))_\theta = \Delta_p(d^\alpha)$$

with equivalent norms. This is [8, Theorem 1.15]. Note that the von Neumann algebra there should be assumed semifinite. Theorem 1.1 can be obtained by duality from Junge and Parcet's theorem. We prefer, however, to give a direct proof. In fact, we will show a slightly stronger result (see Theorem 3.2 below). Note that the pattern of our arguments still models that of [8]. The two main ingredients are again the boundedness of some special Schur multipliers and Pisier's interpolation theorem on triangular subspaces of Schatten classes. Thus our arguments are very similar to those of [8].

Using Haagerup's reduction theorem (see [6]), we deduce from Theorem 1.1 a similar result for type III von Neumann algebras as in [8]. Let \mathcal{N} be a von Neumann algebra equipped with a normal faithful state ψ . Let $L_p(\mathcal{N})$ be the Haagerup noncommutative L_p -spaces associated with \mathcal{N} (see [5] and [15]). Recall that $L_\infty(\mathcal{N}) = \mathcal{N}$ and $L_1(\mathcal{N})$ coincides with the predual of \mathcal{N} . Let d be the operator in $L_1(\mathcal{N})$ corresponding to ψ . For $1 \leq p < \infty$, consider the injection

$$\iota_p : \mathcal{N} \rightarrow L_p(\mathcal{N}), \quad \iota_p(x) = d^{1/p}x + xd^{1/p}.$$

Note that ι_p is injective and of dense range. ι_1 makes $(\mathcal{N}, L_1(\mathcal{N}))$ into a compatible couple. We then have the following two-sided analogue of Kosaki's interpolation theorem [9].

Corollary 1.2 *Let $1 < p < \infty$. Then*

$$(\mathcal{N}, L_1(\mathcal{N}))_{1/p} = L_p(\mathcal{N})$$

with equivalent norms. More precisely, there exist two positive constants c_p and C_p depending only on p such that

$$c_p \|d^{1/p}x + xd^{1/p}\|_p \leq \|x\|_{(\mathcal{N}, L_1(\mathcal{N}))_{1/p}} \leq C_p \|d^{1/p}x + xd^{1/p}\|_p, \quad \forall x \in \mathcal{N}.$$

2 Schur multipliers

In this section we consider some special Schur multipliers on $\mathbb{B}(\ell_2)$, which will play a key role in the proof of our interpolation theorem. These multipliers are of the type already discussed in [8]. Our presentation is, however, independent of [8]. As usual, the operators in $\mathbb{B}(\ell_2)$ are represented as infinite matrices $x = (x_{ij})_{i,j \geq 1}$ (with respect to the canonical matrix units $\{e_{ij}\}$). Recall that a bounded Schur multiplier on $\mathbb{B}(\ell_2)$ is an infinite matrix $\varphi = (\varphi_{ij})_{i,j \geq 1}$ of complex numbers such that $(\varphi_{ij}x_{ij})_{i,j \geq 1} \in \mathbb{B}(\ell_2)$ for any $(x_{ij})_{i,j \geq 1} \in \mathbb{B}(\ell_2)$. The resulting bounded map $(x_{ij})_{i,j} \mapsto (\varphi_{ij}x_{ij})_{i,j}$ will be also denoted by φ .

It is well known that φ is a bounded Schur multiplier on $\mathbb{B}(\ell_2)$ iff there exists a Hilbert space H and two bounded sequences $(\xi_i), (\eta_i) \subset H$ such that

$$\varphi_{ij} = \langle \xi_i, \eta_j \rangle, \quad \forall i, j \geq 1$$

(see [12, Theorem 5.1]). Moreover, in this case φ is automatically completely bounded and $\|\varphi\| = \|\varphi\|_{cb}$. Recall that the cb-norm $\|\varphi\|_{cb}$ of φ is the norm of the map $\text{id}_{\mathbb{B}(\ell_2)} \otimes \varphi$ on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ (usually one uses the compact operators $\mathbb{K}(\ell_2)$ instead of $\mathbb{B}(\ell_2)$ but it does not make any difference). Let $M_{cb}(\mathbb{B}(\ell_2))$, or simply M_{cb} denote the space of all completely bounded Schur multipliers φ on $\mathbb{B}(\ell_2)$, equipped with the norm $\|\varphi\|_{M_{cb}}$. We then have

$$\|\varphi\|_{M_{cb}} = \inf \left\{ \sup_{i,j} \|\xi_i\| \|\eta_j\| : \varphi_{ij} = \langle \xi_i, \eta_j \rangle, \xi_i, \eta_j \in H, H \text{ a Hilbert space} \right\}.$$

The following is a well-known elementary fact (see [8] for a similar statement with more regularity on the function f). We include a proof for the convenience of the reader. As usual, \hat{f} denotes the Fourier transform of a function $f \in L_1(\mathbb{R})$:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(s) e^{-2\pi i \xi s} ds.$$

Proposition 2.1 *Let $f \in L_1(\mathbb{R})$ such that \hat{f} belongs to $L_1(\mathbb{R})$. Then, for any $s_i \in \mathbb{R}$, $(f(s_i - s_j))_{i,j} \in M_{cb}$ and*

$$\|(f(s_i - s_j))_{i,j}\|_{M_{cb}} \leq \|\hat{f}\|_1.$$

In particular, if $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonincreasing convex function, then, for any $s_i \in \mathbb{R}$, $(f(|s_i - s_j|))_{i,j}$ defines a completely positive Schur multiplier on $\mathbb{B}(\ell_2)$.

Proof : By the Fourier inversion formula, we have

$$f(s) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi s} d\xi.$$

Thus letting $g_i(\xi) = \sqrt{|\hat{f}(\xi)|} e^{is_i \xi}$ and $h_j = \overline{\hat{f}(\xi)} / \sqrt{|\hat{f}(\xi)|} e^{is_j \xi}$, we have

$$\|g_i\|_2^2 = \|h_j\|_2^2 = \|\hat{f}\|_1 \quad \text{and} \quad f(s_i - s_j) = \langle g_i, h_j \rangle_{L_2(\mathbb{R})};$$

whence the first assertion.

For the second one, we can suppose $\lim_{t \rightarrow +\infty} f(t) = 0$ since the constant matrix $(1)_{i,j}$ is a completely positive Schur multiplier on $\mathbb{B}(\ell_2)$. Put $g(s) = f(|s|)$ and note that we do not need to assume that $g \in L_1(\mathbb{R})$ to define the Fourier transform (except for $\xi = 0$). Then, it is well known that g is a positive definite function on \mathbb{R} , that is, $\hat{g} \geq 0$ and $\|\hat{g}\|_1 = g(0) = f(0)$, and moreover the Fourier inversion formula holds for g , see [3, Theorem 8.7] for instance. ■

Corollary 2.2 *Let $(\lambda_i)_{i \geq 1}$ be a sequence of positive real numbers. Then, for any $\theta \in [0, 1]$,*

$$\begin{aligned} \left\| \left(\frac{\min(\lambda_i, \lambda_j)}{\max(\lambda_i, \lambda_j)} \right)_{i,j} \right\|_{M_{cb}} &\leq 1, & \left\| \left(\frac{(\lambda_i + \lambda_j)^\theta}{\max(\lambda_i, \lambda_j)^\theta} \right)_{i,j} \right\|_{M_{cb}} &\leq 2^\theta, \\ \left\| \left(\frac{\max(\lambda_i, \lambda_j)^\theta}{(\lambda_i + \lambda_j)^\theta} \right)_{i,j} \right\|_{M_{cb}} &\leq 2 - 2^{-\theta}, & \left\| \left(\frac{\min(\lambda_i, \lambda_j)^\theta}{(\lambda_i + \lambda_j)^\theta} \right)_{i,j} \right\|_{M_{cb}} &\leq 2 - 2^{-\theta}. \end{aligned}$$

Proof : The first multiplier is obtained by applying Proposition 2.1 to the decreasing convex function $f(t) = e^{-t}$. Similarly, the function corresponding to the second multiplier is $f(t) = (1 + e^{-t})^\theta$. To deal with the third one, notice that with $\lambda_i = e^{s_i}$, we have

$$\frac{\max(\lambda_i, \lambda_j)^\theta}{(\lambda_i + \lambda_j)^\theta} = \frac{1}{(1 + e^{-|s_i - s_j|})^\theta} = -\left(1 - \frac{1}{(1 + e^{-|s_i - s_j|})^\theta}\right) + 1.$$

The function $f(t) = 1 - \frac{1}{(1 + e^{-t})^\theta}$ is decreasing and convex on \mathbb{R}_+ , so we get the estimate for the third multiplier. The last one is just the composition of the third with $\left(\frac{\min(\lambda_i^\theta, \lambda_j^\theta)}{\max(\lambda_i^\theta, \lambda_j^\theta)}\right)_{i,j}$, which is a complete contraction. ■

Corollary 2.3 *Let $(\lambda_i)_{i \geq 1}$ and $(\mu_i)_{i \geq 1}$ be two nondecreasing sequences of positive real numbers. Then, for any $\theta \in [0, 1]$,*

$$\left\| \left(\frac{\lambda_i^{1-\theta} \mu_i^\theta + \lambda_j^{1-\theta} \mu_j^\theta}{(\lambda_i + \lambda_j)^{1-\theta} (\mu_i + \mu_j)^\theta} \right)_{i,j} \right\|_{M_{cb}} \leq 9 - 4\sqrt{2}.$$

Proof : Using

$$\lambda_i^{1-\theta} \mu_i^\theta + \lambda_j^{1-\theta} \mu_j^\theta = \max(\lambda_i, \lambda_j)^{1-\theta} \max(\mu_i, \mu_j)^\theta + \min(\lambda_i, \lambda_j)^{1-\theta} \min(\mu_i, \mu_j)^\theta,$$

we immediately get the estimate from Corollary 2.2. ■

Corollary 2.4 *Let $(\lambda_i)_{i \geq 1}$ and $(\mu_i)_{i \geq 1}$ be nondecreasing sequences of positive real numbers. Then, for any $\theta \in [0, 1]$,*

$$\left\| \left(\frac{(\lambda_i + \lambda_j)^\theta (\mu_i + \mu_j)^{1-\theta}}{\lambda_i^\theta \mu_i^{1-\theta} + \lambda_j^\theta \mu_j^{1-\theta}} \right)_{i,j} \right\|_{M_{cb}} \leq 3.$$

Proof : Since $(\lambda_i)_{i \geq 1}$ and $(\mu_i)_{i \geq 1}$ have the same variation, we can write the multiplier under consideration as a composition of three multipliers:

$$\frac{(\lambda_i + \lambda_j)^\theta}{\max(\lambda_i^\theta, \lambda_j^\theta)} \frac{(\mu_i + \mu_j)^{1-\theta}}{\max(\mu_i^{1-\theta}, \mu_j^{1-\theta})} \frac{\max(\lambda_i^\theta \mu_i^{1-\theta}, \lambda_j^\theta \mu_j^{1-\theta})}{\lambda_i^\theta \mu_i^{1-\theta} + \lambda_j^\theta \mu_j^{1-\theta}}.$$

Then the assertion follows from Corollary 2.2. ■

Corollary 2.5 *Let $(\lambda_i)_{i \geq 1}$ be a sequence of positive real numbers. Then, for any $0 < \theta < 1$,*

$$\left\| \left(\frac{\lambda_i^\theta \lambda_j^{1-\theta}}{\lambda_i + \lambda_j} \right)_{i,j} \right\|_{M_{cb}} \leq C \ln \frac{1}{\theta(1-\theta)}.$$

Here, as well as in the sequel, C denotes a universal positive constant.

Proof : According to Proposition 2.1, we have to compute the L_1 -norm of the Fourier transform of $f(s) = (e^{\theta s} + e^{(\theta-1)s})^{-1}$. A standard calculation by the residue theorem yields

$$\hat{f}(\xi) = \frac{\pi}{\sin(\pi(\theta + 2i\pi\xi))}.$$

Then it remains to note that $\|\hat{f}\|_1$ behaves like $\int_0^1 |\theta + ix|^{-1} dx$ when θ is close to 0. ■

Remark 2.6 The preceding corollary is to be compared with [8, Lemma 1.7], which asserts that $\left(\frac{\lambda_i^\theta \lambda_j^{1-\theta}}{\lambda_i + \lambda_j}\right)_{i,j}$ is a bounded multiplier on the triangular subalgebra of $\mathbb{B}(\ell_2)$, for any $0 \leq \theta \leq 1$.

Remark 2.7 Let S_p denote the Schatten p -class, i.e., $S_p = L_p(\mathbb{B}(\ell_2))$ with $\mathbb{B}(\ell_2)$ equipped with the usual trace. Similarly, we define Schur multipliers on S_p as before for $\mathbb{B}(\ell_2)$. It is well-known that any bounded Schur multiplier on $\mathbb{B}(\ell_2)$ is also bounded (even completely bounded) on S_p for any $1 \leq p < \infty$. Let us state this in a slightly more general setting that will be crucial for the next section. Let φ be a bounded Schur multiplier on $\mathbb{B}(\ell_2)$ and \mathcal{M} a von Neumann algebra. Then $\text{id}_{L_p(\mathcal{M})} \otimes \varphi$ defines a bounded map on $L_p(\mathcal{M} \bar{\otimes} \mathbb{B}(\ell_2))$, for any $1 \leq p \leq \infty$:

$$\|(\varphi_{ij} x_{ij})_{i,j}\|_{L_p(\mathcal{M} \bar{\otimes} \mathbb{B}(\ell_2))} \leq \|\varphi\|_{cb} \| (x_{ij})_{i,j} \|_{L_p(\mathcal{M} \bar{\otimes} \mathbb{B}(\ell_2))}$$

for all finite matrices (x_{ij}) with entries in $L_p(\mathcal{M})$.

3 Interpolation

In this section (\mathcal{M}, τ) will denote a semifinite von Neumann algebra and d a density in $L_0(\mathcal{M})$ such that $d^{-1} \in L_0(\mathcal{M})$. For $1 \leq p \leq \infty$, we define

$$L_p(d) = \{x \in L_0(\mathcal{M}) : dx + xd \in L_p(\mathcal{M})\} \quad \text{and} \quad \|x\|_{L_p(d)} = \|dx + xd\|_p.$$

Then $L_p(d)$ is a Banach space. The nontrivial point is the completeness of the norm. This is an immediate consequence of Proposition 3.1 below.

We will use L_d to denote the left multiplication map by d , i.e., $L_d(x) = dx$. Similarly, R_d is the right multiplication map by d . It is clear that both L_d and R_d are continuous on $L_0(\mathcal{M})$. We will also consider them as closed densely defined maps on $L_2(\mathcal{M})$. In this latter case, they are injective, positive and commuting. Thus we can apply functional calculus to them. In particular, $\frac{\sqrt{L_d R_d}}{L_d + R_d}$ is also an injective positive map on $L_2(\mathcal{M})$.

Proposition 3.1 $\frac{\sqrt{L_d R_d}}{L_d + R_d}$ extends to a bounded map on $L_p(\mathcal{M})$, for any $1 \leq p \leq \infty$, with norm $\leq 1/2$. More precisely, we have the following integral representation

$$(3.1) \quad \frac{\sqrt{L_d R_d}}{L_d + R_d}(x) = \int_{\mathbb{R}} u_t(x) \frac{dt}{2 \cosh(\pi t)},$$

where $(u_t)_{t \in \mathbb{R}}$ is the isometry group on $L_p(\mathcal{M})$ defined by

$$u_t(x) = e^{it \ln d} x e^{-it \ln d}.$$

Consequently, $(L_d + R_d)^{-1}$ is a continuous map from $L_p(\mathcal{M})$ to $L_0(\mathcal{M})$ and $L_d + R_d$ is a isometry from $L_p(d)$ onto $L_p(\mathcal{M})$.

Proof : Consider first the case where $d = \sum_{i=1}^k \lambda_i e_i$ for some increasing sequence (λ_i) of positive real numbers and mutually orthogonal projections e_i with sum 1. Note that in this case L_d and R_d are bijections on $L_p(\mathcal{M})$, for any $1 \leq p \leq \infty$. It is also clear that

$$L_d(x) = \sum_i \lambda_i e_i x \quad \text{and} \quad R_d(x) = \sum_j \lambda_j x e_j.$$

Then $\frac{\sqrt{L_d R_d}}{L_d + R_d}$ is given by

$$\frac{\sqrt{L_d R_d}}{L_d + R_d}(x) = \sum_{i,j=1}^k \frac{\sqrt{\lambda_i \lambda_j}}{\lambda_i + \lambda_j} e_i x e_j, \quad \forall x \in L_p(\mathcal{M}).$$

Applying Corollary 2.5 (or its proof) with $\theta = 1/2$ we find

$$\frac{\sqrt{\lambda_i \lambda_j}}{\lambda_i + \lambda_j} = \int_{\mathbb{R}} e^{it(\ln \lambda_i - \ln \lambda_j)} \frac{dt}{2 \cosh(\pi t)}.$$

Thus (3.1) follows and

$$\left\| \frac{\sqrt{L_d R_d}}{L_d + R_d} \right\|_{\mathbb{B}(L_p(\mathcal{M}))} \leq \frac{1}{2}.$$

We then deduce the general case by a standard approximation argument (see also step 2 of the proof of Theorem 3.2 below).

For the second part we note that

$$(L_d + R_d)^{-1} = L_d^{-1/2} \frac{\sqrt{L_d R_d}}{L_d + R_d} R_d^{-1/2}.$$

Since the multiplication map by $d^{-1/2}$ from both left and right is continuous from $L_p(\mathcal{M})$ to $L_0(\mathcal{M})$, we obtain the desired continuity of $(L_d + R_d)^{-1}$. \blacksquare

Theorem 3.2 *Let $f_0, f_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be two nondecreasing functions with $f_i(t) > 0$, for $t > 0$. Put $d_0 = f_0(d)$ and $d_1 = f_1(d)$. Let $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$. Set*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad d_\theta = d_0^{1-\theta} d_1^\theta.$$

Assume $1 < p < \infty$. Consider $(L_{p_0}(d_0), L_{p_1}(d_1))$ as a compatible couple by injecting both spaces into $L_0(\mathcal{M})$. Then, for $0 < \theta < 1$, we have

$$(L_{p_0}(d_0), L_{p_1}(d_1))_\theta = L_p(d_\theta).$$

More precisely, for $x \in L_{p_0}(d_0) \cap L_{p_1}(d_1)$

$$(3.2) \quad C_{p'}^{-1} \|x\|_{L_p(d_\theta)} \leq \|x\|_{(L_{p_0}(d_0), L_{p_1}(d_1))_\theta} \leq C_p \|x\|_{L_p(d_\theta)},$$

where p' denotes the conjugate index of p , and where the constant C_p satisfies the following estimate

$$C_p \leq C \max(p, 2) \max(p, p').$$

The proof of Theorem 3.2 will be divided into two steps. The first one deals with the case where d has only point spectrum. This is the main step. The second one is a simple approximation argument.

Step 1: The discrete case. We assume that $d = \sum_{i=1}^k \lambda_i e_i$ for some increasing sequence (λ_i) of positive real numbers and mutually orthogonal projections e_i with sum 1. Then the map

$$\kappa : \sum_{i,j} e_i x e_j \mapsto \sum_{i,j} e_i x e_j \otimes e_{ij}$$

defines an isometry from $L_p(\mathcal{M})$ into $L_p(\mathcal{M} \bar{\otimes} \mathbb{B}(\ell_2))$, for any $1 \leq p \leq \infty$. Moreover, its range is contractively complemented. We will need the triangular projections:

$$T_+(x) = \sum_{j \geq i} e_i x e_j \quad \text{and} \quad T_-(x) = x - T_+(x).$$

Both T_+ and T_- commute with $L_{f(d)}$ and $R_{f(d)}$.

Lemma 3.3 *For any $1 \leq p \leq \infty$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ nondecreasing,*

$$\frac{2}{3} \|T_\pm(x) f(d)\|_p \leq \|T_\pm(x)\|_{L_p(f(d))} \leq 2 \|T_\pm(x) f(d)\|_p.$$

Proof : We have

$$\begin{aligned} T_+(x) f(d) &= \sum_{j \geq i} f(\lambda_j) e_i x e_j = \sum_{j \geq i} \max(f(\lambda_i), f(\lambda_j)) e_i x e_j, \\ f(d) T_+(x) + T_+(x) f(d) &= \sum_{j \geq i} (f(\lambda_i) + f(\lambda_j)) e_i x e_j. \end{aligned}$$

As for any Schur multiplier $\varphi = (\varphi_{ij})$, we have the identity

$$\kappa \left(\sum_{i,j} \varphi_{ij} e_i x e_j \right) = (\text{id} \otimes \varphi) \kappa \left(\sum_{i,j} e_i x e_j \right),$$

we then deduce the estimates on T_+ using Corollary 2.2 and the transference principle in Remark 2.7. \blacksquare

Remark 3.4 The transference principle also shows that the triangular projections are bounded on $L_p(\mathcal{M})$, for any $1 < p < \infty$:

$$\|T_{\pm}(x)\|_p \leq C \max(p, p') \|x\|_p, \quad \forall x \in L_p(\mathcal{M})$$

for the norms of the triangular projections on S_p are of order $\max(p, p')$.

We will use Pisier's interpolation theorem on subspaces of triangular matrices in [11]. Let $T_p(\mathcal{M}) \subset L_p(\mathcal{M} \otimes \mathbb{B}(\ell_2))$ be the subspace of upper triangular matrices with respect to the matrix units $\{e_{ij}\}$ of $\mathbb{B}(\ell_2)$. For any p_0 and p_1 , the couple $(T_{p_0}(\mathcal{M}), T_{p_1}(\mathcal{M}))$ is compatible in a natural way.

Lemma 3.5 (Pisier) We have

$$(T_{p_0}(\mathcal{M}), T_{p_1}(\mathcal{M}))_{\theta} = T_p(\mathcal{M})$$

with equivalent norms. More precisely, for any $x \in T_{p_0}(\mathcal{M}) \cap T_{p_1}(\mathcal{M})$

$$\|x\|_p \leq \|x\|_{(T_{p_0}(\mathcal{M}), T_{p_1}(\mathcal{M}))_{\theta}} \leq t_p \|x\|_p,$$

where the constant t_p is estimated by $t_p \leq C \max(p, 2)$.

Remark 3.6 The estimate above of t_p is not explicitly stated in [11]. It can be, however, tracked from Pisier's proof. First, by [2, Theorem 3.3.1] and [7, Theorem 4.3], we find

$$t_p \leq C \max(p, p').$$

Next, to see that t_p remains bounded when $p \rightarrow 1$, we use the reiteration theorem and the fact that

$$(T_1(\mathcal{M}), T_2(\mathcal{M}))_{\eta} = T_q(\mathcal{M})$$

holds isomorphically with universal constants, for any $0 < \eta < 1$, where $\frac{1}{q} = 1 - \frac{\eta}{2}$. The latter fact is proved by using the “square argument” of [11] and the Riesz type factorization for triangular matrices (see [16] for more details).

We can now proceed to the second inequality of (3.2). Let $x \in L_{p_0}(d_0) \cap L_{p_1}(d_1)$ with $\|x\|_{L_p(d_{\theta})} < 1$. Thanks to Remark 3.4, we have

$$\|T_+(x)d_{\theta}\|_p < C \max(p, p').$$

As usual, let $\Delta = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$. We now appeal to Lemma 3.5. By basic facts on complex interpolation we find a continuous function $F : \overline{\Delta} \rightarrow T_{p_0}(\mathcal{M}) \cap T_{p_1}(\mathcal{M})$, which is holomorphic in Δ and such that

$$F(\theta) = T_+(x)d_{\theta} \quad \text{and} \quad \sup_{t \in \mathbb{R}} \{ \|F(it)\|_{p_0}, \|F(1+it)\|_{p_1} \} \leq C t_p \max(p, p') \stackrel{\text{def}}{=} C_p.$$

Put $G(z) = F(z)d_0^{z-1}d_1^{-z}$. Since d is discrete and bounded with bounded inverse, G takes its values in $T_{p_0}(\mathcal{M}) \cap T_{p_1}(\mathcal{M})$, is continuous on $\overline{\Delta}$ and holomorphic in Δ . We have $G(\theta) = T_+(x)$. On the other hand, by Lemma 3.3, for $t \in \mathbb{R}$

$$\|G(it)\|_{L_{p_0}(d_0)} = \|T_+(G(it))\|_{L_{p_0}(d_0)} \leq 2\|T_+(G(it))d_0\|_{p_0} = 2\|F(it)d_0^{it}d_1^{-it}\|_{p_0} \leq 2C_p.$$

Similarly,

$$\|G(1+it)\|_{L_{p_1}(d_1)} \leq 2C_p.$$

It follows that

$$\|T_+(x)\|_{(L_{p_0}(d_0), L_{p_1}(d_1))_{\theta}} \leq 2C_p.$$

Arguing in the same way for $T_-(x)$, we get the second inequality of (3.2).

As for the duality of L_p -spaces, the other inequality is obtained by duality. Applying the second inequality of (3.2) to p'_1, p'_0 and $1 - \theta$ instead of p_0, p_1 and θ respectively, we see that the identity

$$\iota : L_{p'}(d_{1-\theta}) \rightarrow (L_{p'_1}(d_0), L_{p'_0}(d_1))_{1-\theta}$$

is bounded. We will dualize this inclusion. The difficulty here lies on the identifications.

First, we reformulate the previous result in terms of non-weighted L_p -spaces. As $d = \sum_{i=1}^k \lambda_i e_i$, the map $\Sigma_d = L_d + R_d$ is a bijection on $L_q(d)$, for any $1 \leq q \leq \infty$. By definition Σ_d is an isometry from $L_q(d)$ onto $L_q(\mathcal{M})$. With this in mind, we can view the compatible couple $(L_{p'_1}(d_0), L_{p'_0}(d_1))$ as $(L_{p'_1}(\mathcal{M}), L_{p'_0}(\mathcal{M}))^t$ via a twisted identification coming from the map

$$t = \Sigma_{d_0}^{-1} \Sigma_{d_1} : L_{p'_1}(\mathcal{M}) \rightarrow L_{p'_0}(\mathcal{M}).$$

Then the maps

$$\begin{aligned} V_0 &= \Sigma_{d_{1-\theta}}^{-1} \Sigma_{d_0} : L_{p'}(\mathcal{M}) \rightarrow L_{p'_1}(\mathcal{M}) \\ V_1 &= \Sigma_{d_{1-\theta}}^{-1} \Sigma_{d_1} : L_{p'}(\mathcal{M}) \rightarrow L_{p'_0}(\mathcal{M}) \end{aligned}$$

are compatible with respect to $(L_{p'_1}(\mathcal{M}), L_{p'_0}(\mathcal{M}))^t$ (i.e., $t \circ V_0 = V_1$), so by interpolation they extend to a bounded map

$$V : L_{p'}(\mathcal{M}) \rightarrow (L_{p'_1}(\mathcal{M}), L_{p'_0}(\mathcal{M}))_{1-\theta}^t.$$

By duality we find

$$V^* : (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))^{t^*, \theta} = (L_{p_1}(\mathcal{M}), L_{p_0}(\mathcal{M}))^{t^*, 1-\theta} \rightarrow L_p(\mathcal{M}).$$

Since

$$(X_0, X_1)_\theta \subset (X_0, X_1)^\theta$$

isometrically for any compatible couple (X_0, X_1) of Banach spaces (see [1]), we can restrict V^* to $(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta^{t^*}$:

$$V^* : (L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_\theta^{t^*} \rightarrow L_p(\mathcal{M}).$$

On the other hand, by duality, the compatibility of the couple $(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))^{t^*}$ comes from the map

$$\Sigma_{d_0}^{-1} \Sigma_{d_1} : L_{p_0}(\mathcal{M}) \rightarrow L_{p_1}(\mathcal{M}).$$

This is due to the fact that all Σ_{d_i} are selfadjoint on $L_2(\mathcal{M})$. Thus $t^* = t$ formally. Also note that V^* is the extension of the compatible maps

$$\begin{aligned} V_1^* &= \Sigma_{d_{1-\theta}}^{-1} \Sigma_{d_1} : L_{p_0}(\mathcal{M}) \rightarrow L_p(\mathcal{M}), \\ V_0^* &= \Sigma_{d_{1-\theta}}^{-1} \Sigma_{d_0} : L_{p_1}(\mathcal{M}) \rightarrow L_p(\mathcal{M}). \end{aligned}$$

Now we return back to the compatible couple $(L_{p_0}(d_0), L_{p_1}(d_1))$. Then note that the maps

$$\Sigma_{d_0} : L_{p_0}(d_0) \rightarrow L_{p_0}(\mathcal{M}) \quad \text{and} \quad \Sigma_{d_1} : L_{p_1}(d_1) \rightarrow L_{p_1}(\mathcal{M})$$

are compatible isometries. Composing them with V^* , we get a bounded map from the interpolated space $(L_{p_0}(d_0), L_{p_1}(d_1))_\theta$ to $L_p(\mathcal{M})$, which extends the following compatible maps

$$\begin{aligned} \Sigma_{d_0} \Sigma_{d_1} \Sigma_{d_{1-\theta}}^{-1} &: L_{p_0}(d_0) \rightarrow L_p(\mathcal{M}), \\ \Sigma_{d_0} \Sigma_{d_1} \Sigma_{d_{1-\theta}}^{-1} &: L_{p_1}(d_1) \rightarrow L_p(\mathcal{M}). \end{aligned}$$

Next, composing the last resulting map with the isometry $\Sigma_{d_\theta}^{-1} : L_p(\mathcal{M}) \rightarrow L_p(d_\theta)$, we deduce that the map

$$\Sigma_{d_0} \Sigma_{d_1} \Sigma_{d_\theta}^{-1} \Sigma_{d_{1-\theta}}^{-1} : (L_{p_0}(d_0), L_{p_1}(d_1))_\theta \rightarrow L_p(d_\theta)$$

is bounded. Namely,

$$(3.3) \quad \|\Sigma_{d_0} \Sigma_{d_1} \Sigma_{d_\theta}^{-1} \Sigma_{d_{1-\theta}}^{-1}(x)\|_{L_p(d_\theta)} \leq C_{p'} \|x\|_{(L_{p_0}(d_0), L_{p_1}(d_1))_\theta}.$$

Finally, to get the first inequality of (3.2), we then just need to correct the left hand side above using Corollary 2.3 for $\Sigma_{d_0}^{-1} \Sigma_{d_1}^{-1} \Sigma_{d_\theta} \Sigma_{d_{1-\theta}}$ corresponds to a bounded Schur multiplier. Thus we obtain the first inequality of (3.2). This finishes the proof of step 1.

Remark 3.7 Alternately, we can also first prove the first inequality of (3.2) as in the appendix of [13], which is essentially an argument dual to the previous one. Then we deduce the second inequality by duality as above.

Step 2: Approximation. Let

$$\mathcal{M}_d = \bigcup_{n \geq 1} q_n(\mathcal{M} \cap L_1(\mathcal{M}))q_n,$$

where $q_n = \chi_{[n^{-1}, n]}(d)$. It is easy to check that \mathcal{M}_d is a dense subspace of $L_p(f(d))$, for $1 \leq p \leq \infty$ (relative to the w^* -topology for $p = \infty$) and for any nondecreasing f on \mathbb{R}_+ . Note that, for any $x \in \mathcal{M}_d$, x belongs to $q_n \mathcal{M} q_n$ for some n . As q_n commutes with d , for such x we have $\|x\|_{L_q(\mathcal{M}, f(d))} = \|x\|_{L_q(q_n \mathcal{M} q_n, f(q_n d))}$. On the other hand, it is clear that $L_q(q_n \mathcal{M} q_n, f(q_n d))$ is a contractively complemented subspace of $L_q(\mathcal{M}, f(d))$. Thus, it is enough to prove the assertion for the reduced algebra $q_n \mathcal{M} q_n$, with $q_n d$ instead of d . Therefore, we can assume that both d and d^{-1} are bounded operators on \mathcal{M} . In this case, $\mathcal{M}_d = L_1(\mathcal{M}) \cap \mathcal{M}$.

Now let (d_n) be a sequence of invertible positive operators with discrete spectrum in the von Neumann subalgebra generated by d such that

$$\|f_i(d_n) - f_i(d)\|_\infty \leq \frac{1}{n}.$$

(For instance, each d_n can be a positive linear combination of mutually orthogonal spectral projections of d .) Then, for any $1 \leq q \leq \infty$, $i = 0, 1$, and $x \in L_1(\mathcal{M}) \cap \mathcal{M}$, we have

$$\lim_n \|x\|_{L_q(f_i(d_n))} = \|x\|_{L_q(f_i(d))}.$$

This is clear as

$$\left| \|x\|_{L_q(f_i(d_n))} - \|x\|_{L_q(f_i(d))} \right| \leq \|(f_i(d_n) - f_i(d))x + x(f_i(d_n) - f_i(d))\|_q \leq \frac{2}{n} \|x\|_q.$$

We go to the interpolation space. Note that $L_1(\mathcal{M}) \cap \mathcal{M}$ is dense in $(L_{p_0}(f_0(d)), L_{p_1}(f_1(d)))_\theta$. Let $x \in L_1(\mathcal{M}) \cap \mathcal{M}$ such that

$$\|x\|_{(L_{p_0}(f_0(d)), L_{p_1}(f_1(d)))_\theta} < 1.$$

Then by [2, Lemma 4.2.3] there exists a function

$$\Psi(z) = \sum_k \psi_k(z) x_k$$

such that

$$\Psi(\theta) = x \quad \text{and} \quad \sup_{t \in \mathbb{R}} \{ \|\Psi(it)\|_{L_{p_0}(f_0(d))}, \|\Psi(1+it)\|_{L_{p_1}(f_1(d))} \} < 1,$$

where (x_k) is a finite sequence in $L_1(\mathcal{M}) \cap \mathcal{M}$ and (ψ_k) a finite sequence of continuous functions on $\overline{\Delta}$, holomorphic in Δ and vanishing at infinity. Using the same function Ψ , but for the couple $(L_{p_0}(f_0(d_n)), L_{p_1}(f_1(d_n)))$, we deduce

$$\|x\|_{(L_{p_0}(f_0(d)), L_{p_1}(f_1(d)))_\theta} \geq \lim_n \|x\|_{(L_{p_0}(f_0(d_n)), L_{p_1}(f_1(d_n)))_\theta}.$$

To get the converse inequality, we again use duality. As $L_1(\mathcal{M}) \cap \mathcal{M}$ is dense in all $L_q(f_i(d))$, all what we need to show is that, for any $y \in L_1(\mathcal{M}) \cap \mathcal{M}$,

$$\lim_n \|y\|_{L_q(f_i(d_n))^*} = \|y\|_{L_q(f_i(d))^*}.$$

However,

$$\|y\|_{L_q(f_i(d_n))^*} = \|(L_{f_i(d)} + R_{f_i(d)})^{-1} y\|_{L_{q'}},$$

and

$$(L_{f_i(d)} + R_{f_i(d)})^{-1} y = \int_0^\infty e^{-f_i(d)t} y e^{-f_i(d)t} dt.$$

Thus the desired result follows from the dominated convergence theorem because $y \in L_1(\mathcal{M}) \cap \mathcal{M}$. Therefore, the proof of Theorem 3.2 is complete.

Remark 3.8 Theorem 3.2 can not be extended to arbitrary positive functions f_0 and f_1 . Otherwise, we would have that the map $\Sigma_{d_0}\Sigma_{d_1}\Sigma_{d_\theta}^{-1}\Sigma_{d_{1-\theta}}^{-1}$ is bounded on $L_p(d_\theta)$ (see (3.3)). In terms of Schur multipliers, this would mean that

$$\left(\frac{(\lambda_i + \lambda_j)(\mu_i + \mu_j)}{(\lambda_i^{1-\theta}\mu_i^\theta + \lambda_j^{1-\theta}\mu_j^\theta)(\lambda_i^\theta\mu_i^{1-\theta} + \lambda_j^\theta\mu_j^{1-\theta})} \right)_{i,j}$$

is a bounded Schur multiplier on S_p , for any positive sequences (λ_i) and (μ_i) , which is false (take $\lambda_i = 1/\mu_i = i$).

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